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ON THE APPORTIONMENT OF REPRESENTATIVES

BY F. W. OWENS, *Cornell University*

The problem of apportionment of representatives has its basis in the concept of the word *representative*, and whom or what he represents. For the purpose of definiteness, the problem will be restricted to the question of apportionment of representatives in the House of Representatives of the United States, except where specified. However, much of the discussion is applicable to any similar apportionment.

As applied to the United States the problem has two interpretations, one taking the question as an abstract one, free from any limitations, constitutional or otherwise, except such as may be deemed just or appropriate on their own merits, and the second treating the problem subject to the limitations imposed by the Constitution. As the latter is the interpretation used in this discussion, it is necessary to note briefly the nature of these restrictions or instructions.

The provisions of the Constitution bearing on the subject are as follows:

> Art. I. Sec. II, 3. Representatives and direct taxes shall be apportioned among the several states which may be included within this Union according to their respective numbers, which shall be determined by adding to the whole number of free persons, including those bound to service for a term of years, and excluding Indians not taxed, three fifths of all other persons. The actual enumeration shall be made within three years after the first meeting of the Congress of the United States, and within every subsequent term of ten years, in such manner as they shall by law direct. The number of representatives shall not exceed one for every thirty thousand, but each State shall have at least one Representative; and until such enumeration shall be made . . .
>
> Sec. IX, 4. No capitation or other direct tax shall be laid, unless in proportion to the census or enumeration hereinbefore directed to be taken.
>
> Amendment. Art. XIV. 2. Representatives shall be apportioned among the several states according to their respective numbers, counting the whole number of persons in each state, excluding Indians not taxed.

Aside from internal contradictions, any method of apportionment must be judged by the completeness with which it satisfies the above provisions. It is at once obvious that if "according to their respective numbers" is accepted as meaning *proportional to*, as it has always been accepted, an apportionment can rarely be made which exactly satisfies the conditions. Various approximations to such an apportionment have been used or suggested. As to the nearness of such approximations, there is no agreed standard of measurement. The language of the apportionment provision being such that exact compliance is impossible, Congress has proceeded to apportion according to what seemed to its members the nearest approach to the intent of the authors of the Constitution. These interpretations themselves have varied from time to time, but the authority of Congress to make such interpretations has never been questioned, probably on account of the provision which makes each House the sole judge of the qualifications of its members.

The first interpretation of Congress was that a number of inhabitants should be chosen arbitrarily and the population of each state should be divided by this number, and the quotient should give the number of representatives to be assigned to the state, *all fractions being neglected.* The method did not cause any disagreement of consequence until after the census of 1830, when a divisor was used which gave to Massachusetts a fraction of 0.99, and Daniel Webster argued that the phrase "according to their respective numbers" should be interpreted as "according to their respective numbers as near as may be," and that a fraction of more than one-half should entitle the state to an extra representative. His contention was not heeded in the apportionment of 1830, but in all later apportionments it has been used as a guiding principle, with minor variations. In 1840 and again in 1910, the method of an assumed ratio, as above, was used, and each state having a fraction greater than one-half was given the extra representative. From 1850 to 1900, a slightly different procedure, known as the Vinton method, was used. It differed from the above method in that first a total size for the House was chosen and the population of the country divided by the assumed size of the House, and this number (the average size of a congressional district) was then used as a divisor of the population of each state, giving for each state a whole number plus a fraction. Representatives were then assigned to the states for all of the whole numbers, and then to the states having the largest fractions, in order of size, until the total number of representatives was brought up to the assumed size of the House. One difficulty found with the method was that sometimes major fractions would not be assigned a representative by this method; in such cases Congress *arbitrarily* assigned extra representatives for such fractions, which varied the method. A lesser difficulty was that states with minor fractions sometimes received an extra representative by the method. This, however, did not seem to meet with any great objection in Congress. Another objection to the method, since known as the Alabama Paradox, is that an increase in the total size of the House may cause a decrease in the number of representatives assigned to some state when this method is used. When Congress found this to occur, a change in the total size of the House was made such that for the new total the paradox did not arise.

The method now known as the method of major fractions was devised by Professor W. F. Willcox, of Cornell University, to correct what appeared to be weaknesses in the Vinton method while preserving the general principle of allowing an extra representative for each major fraction. This was done by going back to the choice of an independent ratio, or size of congressional district, the method differing from that of 1840 only in that a method of choosing the ratio was shown which would yield any desired total size of House and keep the major fraction principle. The method is also free from the Alabama paradox. One great advantage possessed by this method, from the practical standpoint, was that it led to a priority table which showed just the order in which representatives should be assigned to the different states, up to any desired total, provided the method were used. None of the previous methods did this.

Another suggested method, which has never been used by Congress, was devised by Dr. Joseph Hill, of the Bureau of the Census, and called the method of Alternate Ratios. He contended that the extra representative should be given to a state for which the fraction exceeded that for the geometric mean between the preceding and the following whole numbers. If the true ratio (total population divided by total size of House) is used this might involve the Alabama paradox, but an *arbitrary* ratio can always be so chosen that this will not be the case. In devising this method Dr. Hill took the position that there is nothing in the problem analogous to the distribution of money. We might try to analyze this question. The Constitution puts both into the same sentence, saying "Direct taxes and representatives shall be apportioned among the several states according to their respective numbers." This would seem to indicate that the authors of the Constitution considered the problems involved to be equivalent. If one considers them equivalent, he will naturally say that the problem of apportioning any number of representatives, say 435, is equivalent to the problem of dividing a sum of money, say $435,000,000, among the states "according to their respective numbers" subject to the restrictions that the distribution must occur in lump sums of $1,000,000, and so that each state must have at least one share. In order that the share of each inhabitant should be as nearly as possible the same, the method of distribution needs be such that the mathematical expectation* of the resident of each state will be the same as that of the resident of any other state. This will inevitably give rise to some form of the major fraction method, as may be seen as follows: If r is thought of as an independent variable, the unknown population of a state, divided by this r is equally likely to give any remainder from O to $r-1$. If the arithmetic mean is taken as the point of distinction as to whether the extra representative or the extra $1,000,000 is to be assigned, the cases in which the extra assignment is made will just balance those in which it is not made, and the "expectation" of every resident will be the correct number. If the geometric mean were used, however, these cases would not balance, as the geometric mean is less than the arithmetic mean, and the expectation of the resident in the small state would exceed that of the resident of the larger state, as the difference between these means grows less as the numbers become larger. Hence, if the analogy between the distribution of money and the apportionment of representatives is granted, from the standpoint of mathematical expectation, a method based on arithmetic mean should be used.

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* The mathematical expectation is defined as the product of the probability of an event by the reward obtained in case of the event.

The Method of Equal Proportions devised by Professor E. V. Huntington* leads to exactly the same apportionment as that of Dr. Hill, provided that one uses the proper value of r (not the true value) in applying Dr. Hill's process, but his reasoning is considerably different. He uses the four equations

$$\frac{N_i}{P_i}=\frac{N_j}{P_j};\quad \frac{P_i}{N_i}=\frac{P_j}{N_j};\quad \frac{N_i}{N_j}=\frac{P_i}{P_j};\quad \frac{N_j}{N_i}=\frac{P_j}{P_i}$$

as applied to any pair of states. In order to treat these four equations in the same way, he introduces a measure of variation from equality which shows the same variation from equality in all of the four cases. This measure is variously taken as the difference divided by the larger, or by the smaller, or by some average of the two numbers, which quotient gives a sort of *relative* difference. By this criterion and the further postulate that in a satisfactory apportionment there should be no pair of states which is capable of being improved by a transfer of representatives within the pair, such *improvement* being measured by his type of relative error, he is led to the priority table

$$\begin{array}{cccc}
\frac{P_1}{\sqrt{1\cdot 2}}, & \frac{P_1}{\sqrt{2\cdot 3}}, & \frac{P_1}{\sqrt{3\cdot 4}}, & \cdots \\[2ex]
\frac{P_2}{\sqrt{1\cdot 2}}, & \frac{P_2}{\sqrt{2\cdot 3}}, & \frac{P_3}{\sqrt{3\cdot 4}}, & \cdots \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot &
\end{array}$$

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* For details concerning this method see QUART. PUB., Sept. 1921, pp. 859–70.

and he assigns one representative to each state and then the remaining $N-48$ representatives to the states having the $N-48$ largest entries in the table, in the order of their size. As the denominators in the above table are the geometric means between the successive integers, the name "method of the geometric mean" would be appropriate, and the relationship with the method of Dr. Hill is evident. For the rest of this paper this apportionment, without regard to its source, will be called the method H.

The application of the method of least squares was made to this problem for the first time by the present writer.* For a given number of representatives, N, $\frac{N}{P}$ is regarded as the average interest of a resident in a representative. Similarly if N_i is the number of representatives assigned to a state S_i, $\frac{N_i}{P_i}$ will be the actual interest in a representative for each resident in the state S_i. The difference between these numbers represents the injustice done the individual resident of the state, or the error in his assignment. As a measure of the total injustice under a given apportionment of N representatives we may take

$$Q=\sum_{i=1}^{i=48} P_i\left(\frac{N}{P}-\frac{N_i}{P_i}\right)^2.$$

Part of the reasons for squaring the errors lie in the mathematical theory of probability and will not be discussed here. The method has a long standing in the mathematical world, both on account of its theoretical basis in probability and on account of the many useful purposes which it has served. Two strong reasons for the squaring should be at once obvious: all of the terms are thereby made positive, and the effect of over-representing one individual will not then cancel the effect of under-representing another. Also the introduction of large errors is discouraged as the square increases the relative effect of a large error. Without this squaring, even if absolute values were taken, if two individuals were under-represented, to further under-represent one and give the same amount to the other but still leaving him under-represented, would not change the total, and we wish to have every error as well as the total small.

With this basis, the problem is to choose such a distribution of the total number, N, of representatives as will make the total injustice, Q, done to all the residents of the country by the variations in their interest in a representative, as small as possible. That is N_1, N_2, ... N_{48} are to be chosen so that

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* In a paper presented to the American Mathematical Society, February 26, 1921.

$$Q = \sum_{i=1}^{i=48} P_i \left(\frac{N}{P} - \frac{N_i}{P_i} \right)^2$$

is to be as small as possible.

But

$$\begin{aligned} Q &= \Sigma P_i \left(\frac{N^2_i}{P^2_i} - \frac{2NN_i}{PP_i} + \frac{N^2}{P^2} \right) \\ &= \Sigma \frac{N^2_i}{P_i} - 2\Sigma \frac{N}{P} N_i + \frac{N^2}{P^2} \Sigma P_i \\ &= \Sigma \frac{N^2_i}{P_i} - 2\frac{N}{P} \cdot N + \frac{N^2}{P^2} \cdot P \\ &= \Sigma \frac{N^2_i}{P_i} - \frac{N^2}{P}. \end{aligned}$$

Since the last term is a constant, Q is least if

$$Q' = \Sigma \frac{N^2_i}{P_i} \text{ is least.}$$

But

$$Q' = \sum_{i=1}^{i=48} \frac{(1+3+5+ \quad . \ . \ . \quad \overline{2N_i - 1})}{P_i}$$

since

$$1+3+5+, \quad . \ . \ . \quad + \overline{2N_i - 1} = N^2_i.$$

Therefore

$$Q' = 2 \sum_{i=1}^{i=48} \left(\frac{.5}{P_i} + \frac{1.5}{P_i} + \frac{2.5}{P_i} + \quad . \ . \ . \quad + \frac{N_i - \frac{1}{2}}{P_i} \right)$$

must be least.

If we form the table

$$\begin{array}{ccccc} \frac{.5}{P_1}, & \frac{1.5}{P_1}, & \frac{2.5}{P_1}, & \frac{3.5}{P_1}, & . \ . \ . \\ \frac{.5}{P_2}, & \frac{1.5}{P_2}, & \frac{2.5}{P_2}, & \frac{3.5}{P_2}, & . \ . \ . \\ & . & . & . & \\ . & . & . & . & \\ \frac{.5}{P_{48}}, & \frac{1.5}{P_{48}}, & \frac{2.5}{P_{48}}, & \frac{3.5}{P_{48}}, & . \ . \ . \end{array}$$

and choose the N smallest entries in this table and use as N_i the number of entries in the ith line, it is obvious that the quantity Q' will be as small as possible.

If we next form the table

$$\begin{array}{lllll}
\frac{P_1}{.5}, & \frac{P_1}{1.5}, & \frac{P_1}{2.5}, & \frac{P_1}{3.5}, & \cdots \\
\frac{P_2}{.5}, & \frac{P_2}{1.5}, & \frac{P_2}{2.5}, & \frac{P_3}{3.5}, & \cdots \\
\cdot & \cdot & & & \\
\cdot & \cdot & & & \\
\frac{P_{48}}{.5}, & \frac{P_{48}}{1.5}, & \frac{P_{48}}{2.5}, & \frac{P_{48}}{3.5}, & \cdots
\end{array}$$

by using the reciprocals of the entries in the previous table, and choose the N largest entries in this new table, using as N_i the number of entries chosen in the ith line, we have exactly the same distribution as before. But this second form of this priority table shows the method identical with that of major fractions proposed by Professor Willcox. The method of minimizing the sum of the squares of the errors in the interest of a resident in a representative therefore gives another logical basis for this method. For the remainder of this paper this method, as a method of apportionment, irrespective of its logical basis, will be referred to as method W. A special advantage of a criterion for the amount of injustice involved in an apportionment lies in a possible comparison of the amount of injustice involved in apportionment for different total sizes of the House. For this purpose we define

$$Q_1 = N\sqrt{\Sigma P_i\left(\frac{N}{P} - \frac{N_i}{P_i}\right)^2}$$

as the total injustice, as a function of N. If we apply this definition for the populations as they are, we find that a total size of about* 385 gives less injustice than either larger or smaller values of N for a wide range of values.†

The application of the method of Least Squares to the expression $\Sigma\left(\frac{N_i}{N} - \frac{P_i}{P}\right)^2$ will lead to the Vinton method, as was shown by the present writer in March, 1921.‡ The proof is similar to that above and will be omitted.

*** The exact arithmetic involved in this has not been carried out in complete detail.**

† The idea of comparing the total amount of injustice of one apportionment as compared to that for another was first suggested by the present writer in January, 1921.

‡ Presented to the American Mathematical Society, April, 1921.

The application of Least Squares to the less significant ratios, $\frac{P}{N}$ and $\frac{P_i}{N_i}$, in the form

$$\Sigma N_i\left(\frac{P}{N}-\frac{P_i}{N_i}\right)^2$$

will minimize what might be termed the injustice to the representatives due to their having too many or too few constituents. That this method will lead to the method H was proved independently by Professor Huntington and the writer. Still other least square methods have been studied recently by Professor Huntington,[*] in some of which he introduces his form of relative error into the least square method, and with this form the types of expression which lead to methods H and W are interchanged.

Hence, methods H and W might perhaps be said to be the best examples of their respective types. Each is free from the Alabama paradox, and each takes care of the constitutional provision that each state must have at least one representative by assigning one representative to each state first,[†] arbitrarily. Each satisfies the conditions of the Constitution best when judged by its own criteria.

We must, then, in making a choice between these methods use other criteria than those contained in the methods themselves. For this purpose it is useful to examine further the implications of the phrase "among the several states according to their respective numbers." If, as is generally conceded, this means "proportional to," it carries with it the equalities,

$$(1)\ \frac{N}{P}=\frac{N_i}{N_i};\quad (2)\ \frac{P}{N}=\frac{P_i}{N_i};\quad (3)\ \frac{P}{P_i}=\frac{N}{N_i};\quad (4)\ \frac{P_i}{P}=\frac{N_i}{N};$$

$$(5)\ \frac{N}{P}=\frac{\Sigma N_i}{\Sigma P_i};\quad (6)\ \frac{P}{N}=\frac{\Sigma P_i}{\Sigma N_i}\quad (7)\ \frac{P}{\Sigma P_i}=\frac{N}{\Sigma N_i};\quad (8)\ \frac{\Sigma P_i}{P}=\frac{\Sigma N_i}{N}.$$

All of the equations (1), (2), (3), (4), (5), (6), (7), (8) are consequences of the idea of proportionality, and will hold strictly in an exact apportionment. When integers are used they become, in general, inequalities, and for a good apportionment all the variations from equality should be small. Their deviations from equality, however, are not all of the same magnitude. Method W is derived from (1) as the most important of (1), (2), (3), (4), and in the ordinary sense makes (1) more nearly true than does method H. It also considers (5) especially important, and satisfies (5), (6), (7), (8) better than method H does, *no matter which criteria* are used. Although, in terms of its own criteria, method H satisfies (1), (2), (3), (4) better than method W does, it does not satisfy (5), (6), (7), (8) even closely. That this is the case can best be exhibited by specific illustration.

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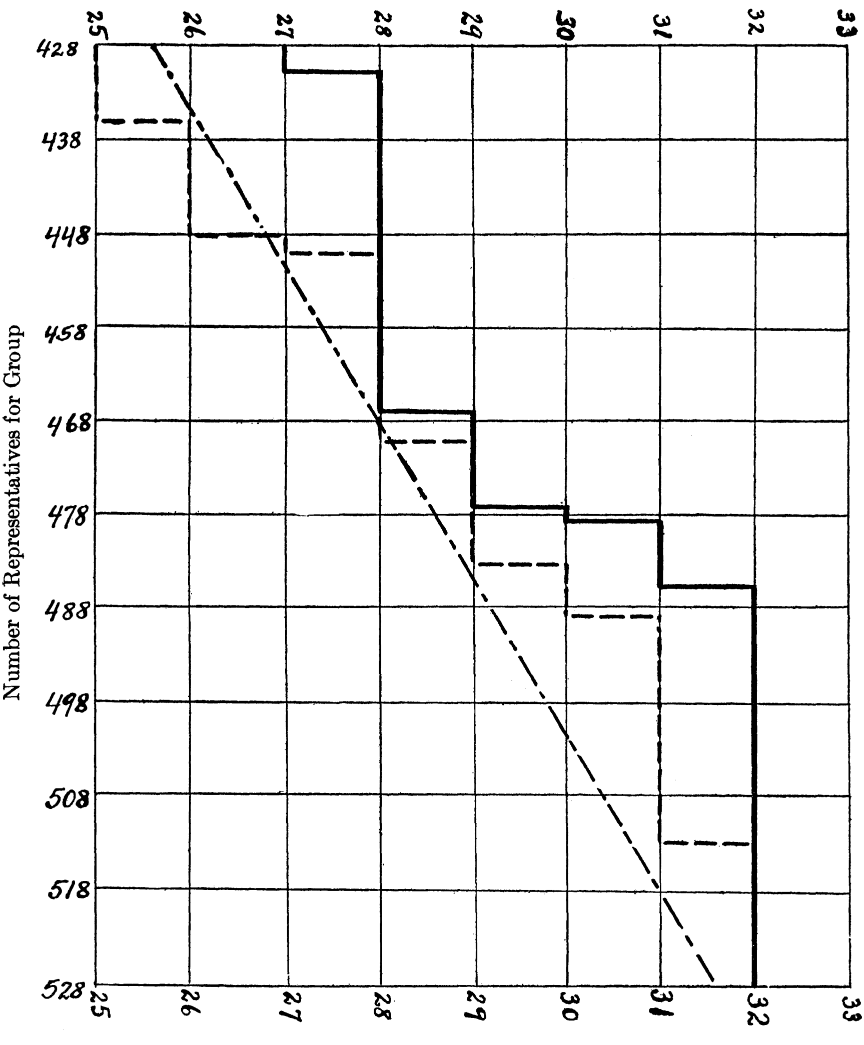
[*] Presented to the American Mathematical Society, September, 1921.

[†] If this requirement were removed the priority table for W would still serve for an analogous method, while the use of the priority table for H would lead to slightly different results according as the error is taken relative to the smaller number, the larger number, or some average of the two.

We will study the effect of apportionments H and W on the representation of a group of small states, consisting of Arizona, Vermont, New Mexico, Idaho, New Hampshire, Utah, Montana, Rhode Island, South Dakota, North Dakota, Maine, Oregon. These are all of the states having a population of as much as 250,000, and less than 800,000. The three still smaller states, Nevada, Wyoming, and Delaware, and their three representatives are omitted from the group on account of the fact that their populations are less than the size of the average congressional district, and hence their representatives are provided by special consitutional provision. The total population of this group of states is 6,310,819. This population multiplied by the total size of House (excluding 3) and divided by the total population (excluding Nevada, Wyoming, and Delaware) will give the true number of representatives to which this group of states is entitled. Under any *fair* method of apportionment the number of representatives assigned to the group would be sometimes more and sometimes less than the true number to which they are entitled when different totals are used for the size of the house, but the discrepancies would tend to average. In making the accompanying diagram, each total size of House from 428 to 527 inclusive was used, thus giving 100 apportionments including all the sizes which have been seriously considered by Congress. For these 100 apportionments, the true (fractional) number of representatives due the group of states is shown by the slanting long and short dash line. The heavy line shows the number assigned to the group by method H. The dash line shows the number assigned to the group by method W. The results may be summarized by saying that by method W the group is under-represented in 22 apportionments and over-represented in the remaining 78 apportionments. The average *amount* of over-representation being 0.47 of one representative or about 1.6 per cent of that due to the group. By method H the group is over-represented in every single one of the 100 apportionments, and the average amount of over-representation is more than 1.4 representatives, or about 5 per cent of the true number due to the group. Hence, if method H were to be used, the group of small states would be *certain* to be over-represented, no matter what the total number of the House, from 428 to 527. This over-representation is necessarily at the expense of under-representing the larger states. This constant



tendency to over-represent the small states (as a group) is inherent in method H and not an accident. The slight over-representation of the group by method W is an accident due to the small range of sizes chosen.

The only manner in which an apportionment can be made that will preserve the equal weight of each individual is by changing to a system whereby each representative would vote according to the number of his constituents. For example, a representative of a district containing

225,000 inhabitants would cast 225,000 votes on every roll call, the representative of a district containing 270,000 inhabitants would cast 270,000 votes, etc. This would seem at first sight to be a cumbrous method, but by installing electrical push buttons and automatic recording and adding devices (an innovation that should be installed in any case to save the waste of time in roll calls), the whole vote would be known instantly on any ballot. This, the only possible way of getting an *exactly* equal representation for each resident, could be carried out only by a change in the language, though not in the spirit, of the Constitution.

CONCLUSION

The question involved in the apportionment of representation is primarily one of constitutional law. The rôle of mathematics in the problem is to make clear the consequences of any given interpretation of the Constitution.

That there is an analogy between the distribution of money and the apportionment of representatives is indicated by their being grouped in the same clause of the Constitution, and on the basis of this analogy method W is to be preferred to method H.

If the intent of the Constitution is that any given population is to have the same expectancy of representation in the House as any other equal population, method H must be abandoned at once. Method W, however, fulfils this condition of equal expectancy much more nearly than does method H, and on this basis also is to be preferred.